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# Heat exchangers and linear image processing theory

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**Abstract**—This paper shows that the transient analysis of some heat exchangers can be derived easily with the linear equations of image processing theory. Partial differential equations of the cross-flow, parallel-flow and rotary heat exchangers are considered together with the corresponding discrete models for linear image processing. Some numerical examples show that the nature of the heat and/or mass transfer problems is similar to those of image processing.

## 1. INTRODUCTION

THE AIM of this paper is to deal with new solutions of the transient response of cross-flow, parallel-flow and rotary heat exchangers [1–13]. It is also our intention to show equivalence between the two-dimensional (2-D) linear image processing models and the models arising from the finite difference approximations of the continuous equations of heat exchangers. An elegant mathematical formulation of the theory of 2-D linear systems is particularly advantageous when one wants to study such important problems of the heat exchangers as optimal control, real-time control, computer-aided design control, parameter identification, etc.

The paper by Baclic and Heggs [1] gives a review of various equivalent forms of the analytical solution for Nusselt's model of the cross-flow heat exchanger. This model has the following general form:

$$\begin{bmatrix} \dfrac{\partial u_1(x,y)}{\partial x} \\ \dfrac{\partial u_2(x,y)}{\partial y} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u_1(x,y) \\ u_2(x,y) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} f(x,y) \quad (1)$$

with given boundary conditions

$$u_1(0,y) = \varphi_1(y), \quad u_2(x,0) = \varphi_2(x). \quad (2)$$

Various mathematical tools have been applied to find the exact solution of equation (1). The methods of the Laplace transform, the Volterra integral equations, successive approximations and the special function techniques are perhaps the best known. The obtained solutions are in the form of an infinite series and need rather difficult and complicated mathematical manipulations. This is also true for parallel-flow [5, 11] and rotary heat exchangers [7].

Therefore, we shall not deal with the continuous models, but propose here special discrete models. All continuous models of the above mentioned heat exchangers can be embedded into the so-called 2-D Roesser and/or Fornasini–Marchesini discrete models of linear image processing theory [14–16].

A coherent theory of the 2-D linear systems gives many possibilities for further exploration, analysis and synthesis of the heat exchangers in terms of 2-D eigenvalues, extended Sylvester and Cayley–Hamilton theorems, 2-D Lyapunov and Riccati equations, etc.

The fundamental paper by Roesser [14] gives and examines the properties of the following discrete image processing model:

$$\begin{bmatrix} u_1(i+1,j) \\ u_2(i,j+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} u_1(i,j) \\ u_2(i,j) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} f(i,j) \quad (3)$$

where $u_1(i,j)$ and $u_2(i,j)$ are horizontal and vertical vectors of $n_1$ and $n_2$ components, respectively, $f(i,j)$ is an input vector of $m$ components, $A_k$ $(k = 1,2,3,4)$ and $B_l$ $(l = 1,2)$ are constant matrices of the appropriate dimensions and $i, j = 0, 1, 2, \ldots$

The Roesser model (equation (3)) can be written in a more compact form

$$u' = Au + Bf \quad (4a)$$

where

$$u' \triangleq [u_1^{\mathrm{T}}(i+1,j),\ u_2^{\mathrm{T}}(i,j+1)]^{\mathrm{T}}$$
$$u = [u_1^{\mathrm{T}}(i,j),\ u_2^{\mathrm{T}}(i,j)]^{\mathrm{T}}. \quad (4b)$$

It is assumed that the boundary conditions $u_1(0,j)$

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## NOMENCLATURE

- $A^{i,j}$, $T^{k,l}_{i,j}$ transition matrices for the Roesser and Fornasini–Marchesini models, respectively
- $A, B, E$ square matrices, $\det E = 0$
- $C$ matrix-to-fluid capacity rate ratio for rotary exchanger, also fluid-to-fluid capacity rate ratio for cross-flow exchanger, equation (16)
- $c$ parameter, equation (44), $a^2\Delta t/\Delta x^2$
- $f$ two-dimensional input function, equations (1) and (3)
- $I$ unit matrix
- $I, J$ upper limits for $i$ and $j$, respectively, equation (22b)
- $i$ discrete spatial variable
- $j$ discrete spatial or time variable
- $K_1, K_2$ parameters, equations (18) and (24)
- $N_{tu}$ number of transfer units, equations (16), (22b) and (28)
- $t$ dimensionless time variable
- $U$ dimensionless matrix temperature of rotary exchanger, also dimensionless wall temperature of parallel exchanger
- $u$ dimensionless fluid temperature
- $W$ thermal capacity rate [W K$^{-1}$]
- $x, y$ dimensionless continuous spatial variables
- $\Delta x, \Delta y, \Delta t$ net spacing for $x$, $y$ and $t$, respectively
- $Y, F$ two-dimensional $Z$ transforms, equation (47)
- $z_1, z_2$ complex variables, equation (39b).

Greek symbols

- $\alpha$ parameter, equation (43)
- $\beta_n$ dimensionless duration of period $n$ of rotary exchanger
- $\beta_s$ dimensionless azimuthal position of change in non-uniform temperature distribution of Fig. 7
- $\gamma, \beta$ parameters in initial and boundary conditions, equations (24) and (27)
- $\delta$ parameter describing the temperature distribution in Fig. 7
- $\varepsilon$ effectiveness of the cross-flow exchanger, equation (16)
- $\phi, \psi, \varphi$ boundary and/or initial conditions
- $\omega$ thermal capacity rate ratio, equation (24).

Subscripts

- $i, j$ integers of the transition matrix of model with varying coefficients
- m mean integral temperature
- min minimum value, equation (22b)
- $n$ period of operation, 1, 2 (rotary exchanger)
- $n_1, n_2$ dimensions of horizontal and vertical vectors in Roesser model
- w wall
- 0 Roesser and Fornasini–Marchesini models
- 1 weaker fluid flowing in $x$-direction (cross-flow exchanger), also refers to fluid flowing in period 1 (rotary exchanger), and to Roesser and Fornasini–Marchesini models
- 2 stronger fluid flowing in $y$-direction (cross-flow exchanger), also refers to fluid flowing in period 2 (rotary exchanger), and to Roesser and Fornasini–Marchesini models
- 3, 4 matrices of the Roesser model.

Superscripts

- $i, j, (k, l)$ integers of the transition matrix of the model with constant (varying) coefficients
- shift operator for Roesser model, equations (4) and (39).

and $u_2(i, 0)$ are known. Then the solution of equation (3) takes the form [14]

$$\begin{bmatrix} u_1(i,j) \\ u_2(i,j) \end{bmatrix} = \sum_{\alpha=0}^{j} A^{i,j-\alpha} \begin{bmatrix} u_1(0,\alpha) \\ 0 \end{bmatrix} + \sum_{\beta=0}^{i} A^{i-\beta,j} \begin{bmatrix} 0 \\ u_2(\beta,0) \end{bmatrix} + \sum_{\alpha=0}^{i} \sum_{\beta=0}^{j} \left\{ A^{i-\alpha-1,j-\beta} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + A^{i-\alpha,j-\beta-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \right\} f(\alpha,\beta) \quad (5)$$

where the transition matrix $A^{i,j}$ is given by the recursive formulas

$$A^{i,j} = A^{1,0}A^{i-1,j} + A^{0,1}A^{i,j-1}$$

$$A^{1,0} = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}, \quad A^{0,1} = \begin{bmatrix} 0 & 0 \\ A_3 & A_4 \end{bmatrix}$$

$$A^{0,0} = I$$

$$A^{-i,j} = A^{i,-j} = 0, \quad i, j = 1, 2, 3, \ldots \quad (6)$$

There are a great number of papers and some books concerning the problems of analysis and synthesis of 2-D systems described via equation (3). One can see for example, books by Eising [17], Kaczorek [18], Bose [19] and papers by Kung *et al.* [20], Stavroulakis and Paraskevopoulos [21], Eising [22, 23], Sebek [24], Tzafestas and Pimenides [25], Mertzios and Lewis [26], Ciftsibasi and Yuksel [27], Lewis and Mertzios [28], Lodge and Fahmy [29], Marszałek [30], Marszałek and Sadecki [31], Marszałek and Kekkeris [32] where the Roesser model was examined with respect to such control properties as controllability, observability, stability, optimal control, exact model matching, decoupling, parameter indentification and others.

Recently, a generalized version of equation (3), so-called singular 2-D Roesser model has been proposed [33, 34]. This model is studied in more detail in Section 4.

In 1976 and 1978 Fornasini and Marchesini proposed two different 2-D state-space discrete models for linear image processing. The so-called first and second models of Fornasini and Marchesini are given respectively as follows:

$$u(i+1,j+1) = A_1u(i,j+1)+A_2u(i+1,j) + A_0u(i,j)+Bf(i,j)$$

$$u(i+1,j+1) = A_1u(i,j+1)+A_2u(i+1,j) + B_1f(i,j+1)+B_2f(i+1,j). \quad (7)$$

The transition matrices for both models can be obtained recursively in a way similar to that of equations (6).

Some generalizations of the 2-D discrete models with constant matrices to models with varying coefficients are given in refs. [35, 36].

In this paper some fundamental properties of the Roesser (equation (1)) and Fornasini–Marchesini (equations (7)) models for linear image processing techniques are given. It is shown that the nature of these models is appropriate for the analysis of various heat exchangers. The analysis is simple, straightforward and easy for computer implementation. This is in great contrast with other analytical methods, which use a lot of complicated [5, 10, 11] formulas and special functions. The advantages of the proposed method can be easily understood from the numerical examples given in the following sections.

## 2. 2-D ROESSER DISCRETE MODEL IN NUMERICAL ANALYSIS OF THE HEAT EXCHANGERS

### 2.1. *Cross-flow heat exchanger*

Consider equation (1) with $a_{11} = -a_{12} = -a_{21} = a_{22} = -1$, $b_1 = b_2 = 0$ and the forward difference quotients for both partial derivatives. Denoting the net spacings for $x$ and $y$ as $\Delta x$ and $\Delta y$, respectively, we have

$$\begin{bmatrix} u_1(i+1,j) \\ u_2(i,j+1) \end{bmatrix} = \begin{bmatrix} 1-\Delta x & \Delta x \\ \Delta y & 1-\Delta y \end{bmatrix} \begin{bmatrix} u_1(x,y) \\ u_2(x,y) \end{bmatrix}. \quad (8)$$

Next, by choosing $u_1(0,j) = 1$, $u_2(i,0) = 0$ we shall prove the discrete version of the well-known equality given by equations (46) and (47) in Baclic and Heggs [1]

$$u_1(x,y)+u_2(y,x) = 1. \quad (9)$$

This equality can be written for $\Delta x = \Delta y$ as

$$u_1(i,j)+u_2(j,i) = 1. \quad (10)$$

The proof can be accomplished using induction with respect to $i$ and $j$, as follows.

(a) For $i = j = 0$ we have from the given boundary conditions

$$u_1(0,0)+u_2(0,0) = 1. \quad (11)$$

(b) Assuming, equation (10) is true for points $(i-1,0)$ and $(0,j-1)$ we shall show that it is valid for points $(i,0)$ and $(0,j)$, i.e. that

$$u_1(i,0)+u_2(0,i) = 1 \quad (12a)$$

$$u_1(0,j)+u_2(j,0) = 1. \quad (12b)$$

Equation (12b) is obvious from the known boundary conditions. Taking into account equation (8) we obtain

$$\begin{aligned} u_1(i,0)+u_2(0,i) &= (1-\Delta x)u_1(i-1,0) \\ &\quad +\Delta x\, u_2(i-1,0) \\ &\quad +\Delta x\, u_1(0,i-1) \\ &\quad +(1-\Delta x)u_2(0,i-1) \\ &= 1+\Delta x-(\Delta x-\Delta x) \\ &\quad \times u_1(i-1,0)-\Delta x \\ &= 1 \end{aligned}$$

where the assumption $u_1(i-1,0)+u_2(0,i-1) = 1$ was used.

(c) Next, assuming

$$u_1(i-1,j)+u_2(j,i-1) = 1 \quad (13a)$$

$$u_1(i,j-1)+u_2(j-1,i) = 1 \quad (13b)$$

we shall show that

$$u_1(i,j)+u_2(j,i) = 1. \quad (14)$$

Really, taking into account equations (8) and (13) we obtain

$$\begin{aligned} u_1(i,j)+u_2(j,i) &= (1-\Delta x)u_1(i-1,j) \\ &\quad +\Delta x\, u_2(i-1,j) \\ &\quad +\Delta x\, u_1(j,i-1) \\ &\quad +(1-\Delta x)u_2(j,i-1) \\ &= u_1(i-1,j)+u_2(j,i-1) \\ &\quad +(\Delta x-\Delta x)[u_1(j,i-1) \\ &\quad +u_1(i-1,j)-1] \\ &= 1. \end{aligned}$$

This completes the proof. If the net spacings are such that $\Delta x \neq \Delta y$, but $\Delta y = \alpha\Delta x$ (or $\Delta x = \alpha\Delta y$), $\alpha$ being an integer, then equation (10) takes the form

$$u_1(i,j)+u_2\left(\alpha j, \frac{i}{\alpha}\right) = 1 \quad \text{for} \quad i = \alpha\beta \qquad (15a)$$

or

$$u_1\left(\alpha j, \frac{i}{\alpha}\right)+u_2(i,j) = 1 \quad \text{for} \quad j = \alpha\beta \qquad (15b)$$

where $\beta$ is an integer.

The discrete model (equation (3)) can be easily applied for analysis of the effectiveness of the heat exchanger defined as [1, 4]

$$\varepsilon = 1-\frac{\Delta y}{CN_{tu}}\sum_{j=0}^{CN_{tu}/\Delta y} u_1\left(\frac{N_{tu}}{\Delta x}, j\right) \qquad (16)$$

upon the condition that $CN_{tu}/\Delta y$ and $N_{tu}/\Delta x$ are some integers which define the discrete upper limits of the rectangular area of the continuous model (equation (1)): $0 \leqslant x \leqslant N_{tu}$, $0 \leqslant y \leqslant CN_{tu}$. Application of formula (16) to the discrete model (equation (8)) yields the values of $\varepsilon$ given in Table 1. These values were obtained by choosing the numbers of subintervals for $x$ and $y$ to be $50\times 50$ for each pair of $C$ and $N_{tu}$. Comparing our results with those given in Table 2 of Baclic and Heggs [1] we have found that the maximum difference is less than 1.5%.

Next, the 2-D Roesser model has been applied for calculations of the mean integral temperatures. One particular result of the calculations is shown in Fig. 1 which presents a joint plot of the effectiveness and the mean integral temperature of the stronger fluid as a function of $C$ and $N_{tu}$. For any given point A with parameters $N_{tuA}$, $C_A$, $\varepsilon_A$ one can easily find the mean integral temperature $u_{1m}$ and the mean integral difference $\Delta u_m$ as described by Baclic and Heggs and shown in Fig. 1 where

$$u_{km} = \frac{\Delta y}{CN_{tu}}\frac{\Delta x}{N_{tu}}\sum_{i=0}^{N_{tu}/\Delta x}\sum_{j=0}^{CN_{tu}/\Delta y} u_k(i,j),$$

$$k = 1,2$$

$$\Delta u_m = u_{1m}-u_{2m}. \qquad (17)$$

The results obtained compare well with those given by Baclic and Heggs [1].

Finally, to end the numerical examples for the steady-state response of the cross-flow heat exchanger the temperature distributions of the stronger and weaker fluids obtained via model (8) for $C = 1$, $N_{tu} = 10$ and $\Delta x = \Delta y = 0.2$ are presented in Figs. 2(a)–(d). These results also compare well with the respective results in Lach and Pieczka (see Figs. 2 and 3 in ref. [3]) which were obtained by using special functions and infinite series (see equation (10) in ref. [2] or equations (3) and (4) in ref. [3]).

The application of the discrete Roesser model is not only restricted to the 2-D case. It is easy to write a general $n$-D Roesser model [37]. If for example the transient analysis of the cross-flow heat exchanger where the temperature of the exchanger core is considered, then a 3-D (two spatial and one time variable) continuous model can be derived [10]

$$\frac{\partial U_w}{\partial t} = K_1(u_1-U_w)-K_2(U_w-u_2)$$

$$K_2\frac{\partial u_1}{\partial x} = -u_1+U_w$$

$$K_1\frac{\partial u_2}{\partial y} = -u_2+U_w \qquad (18)$$

with the following boundary and initial conditions:

$$u_1(0,y,t) = \phi_1(y,t)$$

$$u_2(x,0,t) = \phi_2(x,t)$$

$$U_w(x,y,0) = \psi_w(x,y). \qquad (19)$$

A straightforward finite difference approximation yields

$$\begin{bmatrix} u_1(i+1,j,k) \\ u_2(i,j+1,k) \\ U_w(i,j,k+1) \end{bmatrix} = \begin{bmatrix} 1-\Delta x/K_2 & 0 & \Delta x/K_2 \\ 0 & 1-\Delta y/K_1 & \Delta y/K_1 \\ \Delta t K_1 & \Delta t K_2 & 1-K_1\Delta t-K_2\Delta t \end{bmatrix} \times \begin{bmatrix} u_1(i,j,k) \\ u_2(i,j,k) \\ U_w(i,j,k) \end{bmatrix}. \qquad (20)$$

Thus the model is a 3-D one and the corresponding transition matrix can be recursively computed with the scheme (compare with equation (6))

$$A^{i,j,k} = A^{1,0,0}A^{i-1,j,k}+A^{0,1,0}A^{i,j-1,k}+A^{0,0,1}A^{i,j,k-1}. \qquad (21)$$

Model (20) has been applied for numerical calculations of the mean mixed outlet temperatures defined as follows:

$$u_{1m}(k) = \frac{1}{J}\sum_{j=0}^{J} u_1(I,j,k)$$

$$u_{2m}(k) = \frac{1}{I}\sum_{i=0}^{I} u_2(i,J,k) \qquad (22a)$$

where

$$I = N_{tu}\frac{W_{min}}{W_1\Delta x}$$

$$J = N_{tu}\frac{W_{min}}{W_2\Delta y} \qquad (22b)$$

with the following boundary and initial conditions:

Table 1. Effectiveness of cross-flow heat exchanger as a function of $C$ and $N_{tu}$

| | $C$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $N_{tu}$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| 0.25 | 0.2036 | 0.2012 | 0.1988 | 0.1964 | 0.1941 | 0.1918 | 0.1895 | 0.1873 | 0.1851 | 0.1829 |
| 0.50 | 0.3752 | 0.3677 | 0.3604 | 0.3532 | 0.3462 | 0.3394 | 0.3328 | 0.3263 | 0.3200 | 0.3138 |
| 0.75 | 0.5074 | 0.4944 | 0.4817 | 0.4694 | 0.4575 | 0.4460 | 0.4348 | 0.4239 | 0.4134 | 0.4032 |
| 1.00 | 0.6099 | 0.5919 | 0.5744 | 0.5575 | 0.5412 | 0.5254 | 0.5102 | 0.4955 | 0.4812 | 0.4675 |
| 1.25 | 0.6897 | 0.6677 | 0.6464 | 0.6258 | 0.6058 | 0.5865 | 0.5679 | 0.5500 | 0.5326 | 0.5159 |
| 1.50 | 0.7521 | 0.7272 | 0.7031 | 0.6796 | 0.6568 | 0.6348 | 0.6134 | 0.5928 | 0.5729 | 0.5537 |
| 1.75 | 0.8012 | 0.7745 | 0.7484 | 0.7229 | 0.6980 | 0.6738 | 0.6503 | 0.6275 | 0.6055 | 0.5842 |
| 2.00 | 0.8399 | 0.8124 | 0.7851 | 0.7582 | 0.7318 | 0.7060 | 0.6808 | 0.6562 | 0.6324 | 0.6094 |
| 2.25 | 0.8707 | 0.8430 | 0.8152 | 0.7875 | 0.7601 | 0.7330 | 0.7064 | 0.6805 | 0.6552 | 0.6307 |
| 2.50 | 0.8951 | 0.8679 | 0.8401 | 0.8121 | 0.7839 | 0.7560 | 0.7284 | 0.7013 | 0.6748 | 0.6490 |
| 2.75 | 0.9147 | 0.8884 | 0.8609 | 0.8329 | 0.8044 | 0.7759 | 0.7474 | 0.7194 | 0.6918 | 0.6649 |
| 3.00 | 0.9305 | 0.9052 | 0.8785 | 0.8507 | 0.8221 | 0.7932 | 0.7641 | 0.7353 | 0.7068 | 0.6789 |
| 3.25 | 0.9432 | 0.9193 | 0.8934 | 0.8660 | 0.8376 | 0.8084 | 0.7789 | 0.7494 | 0.7202 | 0.6914 |
| 3.50 | 0.9534 | 0.9310 | 0.9062 | 0.8794 | 0.8512 | 0.8219 | 0.7921 | 0.7621 | 0.7322 | 0.7026 |
| 3.75 | 0.9618 | 0.9409 | 0.9171 | 0.8911 | 0.8632 | 0.8340 | 0.8040 | 0.7735 | 0.7430 | 0.7127 |
| 4.00 | 0.9685 | 0.9492 | 0.9266 | 0.9013 | 0.8739 | 0.8449 | 0.8147 | 0.7839 | 0.7529 | 0.7219 |
| 4.25 | 0.9741 | 0.9562 | 0.9348 | 0.9104 | 0.8835 | 0.8547 | 0.8245 | 0.7934 | 0.7619 | 0.7304 |
| 4.50 | 0.9786 | 0.9622 | 0.9420 | 0.9185 | 0.8922 | 0.8637 | 0.8334 | 0.8021 | 0.7702 | 0.7382 |
| 4.75 | 0.9823 | 0.9673 | 0.9483 | 0.9257 | 0.9000 | 0.8718 | 0.8417 | 0.8102 | 0.7779 | 0.7454 |
| 5.00 | 0.9853 | 0.9717 | 0.9539 | 0.9322 | 0.9071 | 0.8793 | 0.8492 | 0.8176 | 0.7851 | 0.7521 |
| 5.25 | 0.9879 | 0.9754 | 0.9587 | 0.9380 | 0.9136 | 0.8861 | 0.8562 | 0.8245 | 0.7917 | 0.7583 |
| 5.50 | 0.9899 | 0.9787 | 0.9630 | 0.9432 | 0.9195 | 0.8925 | 0.8627 | 0.8310 | 0.7979 | 0.7641 |
| 5.75 | 0.9916 | 0.9814 | 0.9669 | 0.9479 | 0.9249 | 0.8983 | 0.8688 | 0.8370 | 0.8037 | 0.7696 |
| 6.00 | 0.9930 | 0.9838 | 0.9702 | 0.9522 | 0.9298 | 0.9037 | 0.8744 | 0.8427 | 0.8092 | 0.7747 |
| 6.25 | 0.9942 | 0.9859 | 0.9733 | 0.9560 | 0.9344 | 0.9088 | 0.8797 | 0.8480 | 0.8144 | 0.7796 |
| 6.50 | 0.9952 | 0.9877 | 0.9759 | 0.9596 | 0.9386 | 0.9135 | 0.8847 | 0.8530 | 0.8193 | 0.7842 |
| 6.75 | 0.9960 | 0.9893 | 0.9783 | 0.9628 | 0.9425 | 0.9179 | 0.8894 | 0.8578 | 0.8239 | 0.7885 |
| 7.00 | 0.9967 | 0.9906 | 0.9805 | 0.9657 | 0.9461 | 0.9220 | 0.8938 | 0.8623 | 0.8283 | 0.7927 |
| 7.25 | 0.9972 | 0.9918 | 0.9824 | 0.9684 | 0.9495 | 0.9258 | 0.8979 | 0.8665 | 0.8325 | 0.7966 |
| 7.50 | 0.9977 | 0.9928 | 0.9841 | 0.9708 | 0.9526 | 0.9294 | 0.9019 | 0.8706 | 0.8364 | 0.8003 |
| 7.75 | 0.9981 | 0.9937 | 0.9857 | 0.9731 | 0.9554 | 0.9328 | 0.9056 | 0.8744 | 0.8402 | 0.8039 |
| 8.00 | 0.9984 | 0.9945 | 0.9871 | 0.9751 | 0.9581 | 0.9360 | 0.9091 | 0.8781 | 0.8439 | 0.8073 |
| 8.25 | 0.9987 | 0.9952 | 0.9883 | 0.9770 | 0.9606 | 0.9390 | 0.9125 | 0.8816 | 0.8473 | 0.8106 |
| 8.50 | 0.9989 | 0.9958 | 0.9895 | 0.9788 | 0.9630 | 0.9419 | 0.9157 | 0.8850 | 0.8506 | 0.8138 |
| 8.75 | 0.9991 | 0.9963 | 0.9905 | 0.9804 | 0.9652 | 0.9446 | 0.9187 | 0.8882 | 0.8538 | 0.8168 |
| 9.00 | 0.9992 | 0.9968 | 0.9914 | 0.9818 | 0.9672 | 0.9471 | 0.9216 | 0.8912 | 0.8569 | 0.8197 |
| 9.25 | 0.9994 | 0.9972 | 0.9922 | 0.9832 | 0.9691 | 0.9495 | 0.9244 | 0.8942 | 0.8599 | 0.8225 |
| 9.50 | 0.9995 | 0.9975 | 0.9930 | 0.9844 | 0.9709 | 0.9518 | 0.9270 | 0.8970 | 0.8627 | 0.8252 |
| 9.75 | 0.9996 | 0.9978 | 0.9936 | 0.9856 | 0.9726 | 0.9540 | 0.9295 | 0.8997 | 0.8654 | 0.8278 |
| 10.00 | 0.9996 | 0.9981 | 0.9942 | 0.9867 | 0.9742 | 0.9560 | 0.9319 | 0.9024 | 0.8681 | 0.8303 |

$$\phi_1(y,t) = 1-(1-\beta)\exp(-\gamma_1 t)$$
$$\phi_2(x,t) = \beta\exp(-\gamma_2 t)$$
$$\psi_w(x,y) = \beta$$
$$\beta = 0.75,\ \gamma_1 = 2,\ \gamma_2 = 1,\ K_1 = 0.25,$$
$$K_2 = 1-K_1,\ W_1/W_2 = 0.5. \quad (23)$$

The number of discrete points chosen for calculations were 26, 26 and 51 for $x$, $y$ and $t$, respectively. Figure 3 presents the mean mixed outlet temperatures together with $\phi_1(y,t)$ and $\phi_2(x,t)$. These results are exactly the same as an analytical solution given by Gvozdenac (see Fig. 2 of ref. [10]).

2.2. *Parallel-flow heat exchanger*

The 2-D Roesser model can also be easily applied for the analysis of parallel-flow heat exchangers with the continuous model considered by Gvozdenac [11]. This model is as follows:

$$\frac{\partial U_w}{\partial t} + U_w = K_1 u_1 + K_2 u_2$$
$$K_2\frac{\partial u_1}{\partial x} = U_w - u_1$$
$$\frac{K_1}{\omega}\frac{\partial u_2}{\partial x} = U_w - u_2 \quad (24)$$

with the initial and boundary conditions

$$u_1(0,t) = \phi_1(t)$$
$$u_2(0,t) = \phi_2(t)$$
$$U_w(x,0) = \psi_w(x). \quad (25)$$

As pointed out in ref. [11] this model is valid if the thermal capacities of the masses of the two fluids are negligibly small relative to the thermal capacity of the exchanger core. The general analytical solution obtained by Gvozadenac [11] is quite complicated. A

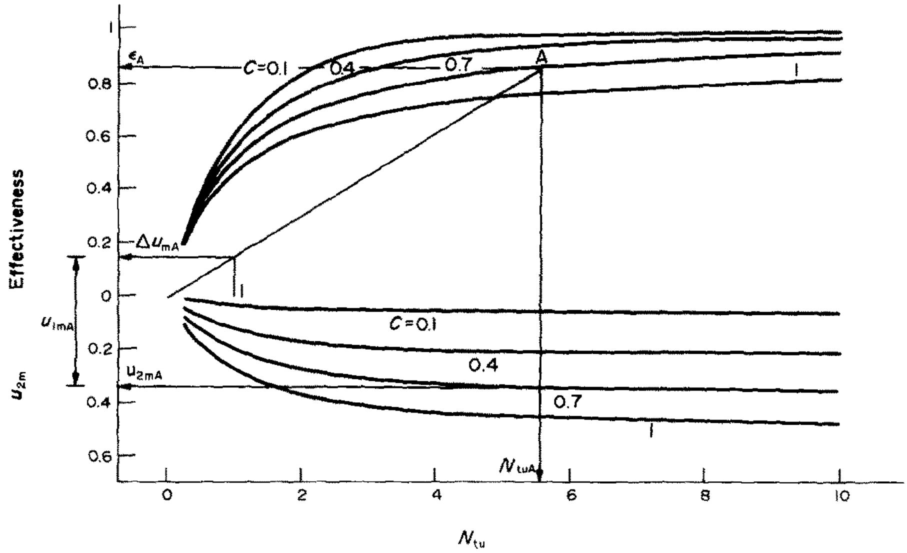


FIG. 1. Joint plot of the effectiveness and the mean integral temperature of the stronger fluid as a function of $C$ and $N_{tu}$.

simple analysis of equations (24) can be performed if one applies the respective difference model as follows:

$$\begin{bmatrix} u_1(i+1,j) \\ u_2(i+1,j) \\ \cdots\cdots \\ U_w(i,j+1) \end{bmatrix} = \left[\begin{array}{cc|c} 1-\dfrac{\Delta x}{K_2} & 0 & \dfrac{\Delta x}{K_2} \\ 0 & 1-\dfrac{\omega\Delta x}{K_1} & \dfrac{\Delta x\omega}{K_1} \\ \hline \Delta t K_1 & \Delta t K_2 & 1-\Delta t \end{array}\right] \begin{bmatrix} u_1(i,j) \\ u_2(i,j) \\ \cdots\cdots \\ U_w(i,j) \end{bmatrix} \tag{26}$$

with the horizontal and vertical vectors of two and one components, respectively.

This model was used for calculations of the outlet temperatures for $x \in (0,4)$, $\omega = 0.5$, $K_1 = 0.25$, $K_2 = 1-K_1$ and the following boundary and initial conditions:

$$\begin{aligned} \phi_1(t) &= 1-(1-\beta)\exp(-\gamma_1 t) \\ \phi_2(t) &= \beta\exp(-\gamma_2 t) \\ \psi_w(x) &= \beta \end{aligned} \tag{27}$$

with $\beta = 0.75$, $\gamma_1 = 2$, $\gamma_2 = 1$.

The results obtained for $\Delta x = 0.08$ and $\Delta t = 0.4$ are shown in Fig. 4. They also compare well with Gvozdenac's results (see Fig. 2 of ref. [11]).

### 2.3. *Rotary heat exchanger*

As it was shown above the application of Roesser's model is possible not only for problems with spatial variables but also for variables of mixed nature. The next example illustrates the transformation of the continuous model of the rotary heat exchanger into its respective 2-D Roesser model.

The rotary heat exchanger can be described with the following equations [7]:

$$\left.\begin{aligned} \frac{\partial u_n}{\partial x} + \beta_n C_n \frac{\partial U}{\partial t} &= 0 \\ \beta_n C_n \frac{\partial U}{\partial t} + N_{tun}(U-u_n) &= 0 \end{aligned}\right\} n = 1,2. \tag{28}$$

A slight modification of equations (28) gives

$$\begin{bmatrix} \dfrac{\partial u_n}{\partial x} \\ \dfrac{\partial U}{\partial t} \end{bmatrix} = \begin{bmatrix} -N_{tun} & N_{tun} \\ \dfrac{N_{tun}}{\beta_n C_n} & -\dfrac{N_{tun}}{\beta_n C_n} \end{bmatrix} \begin{bmatrix} u_n \\ U \end{bmatrix} \tag{29}$$

and the corresponding 2-D discrete model is

$$\begin{bmatrix} u_n(i+1,j) \\ U(i,j+1) \end{bmatrix} = \begin{bmatrix} 1-\Delta x\, N_{tun} & \Delta x\, N_{tun} \\ \dfrac{\Delta t\, N_{tun}}{\beta_n C_n} & 1-\dfrac{\Delta t\, N_{tun}}{\beta_n C_n} \end{bmatrix} \begin{bmatrix} u_n(i,j) \\ U(i,j) \end{bmatrix}. \tag{30}$$

This model can be used for similar computational examples as done by Brandemuehl and Banks [7]. Two samples of the examples are shown in Figs. 5 and

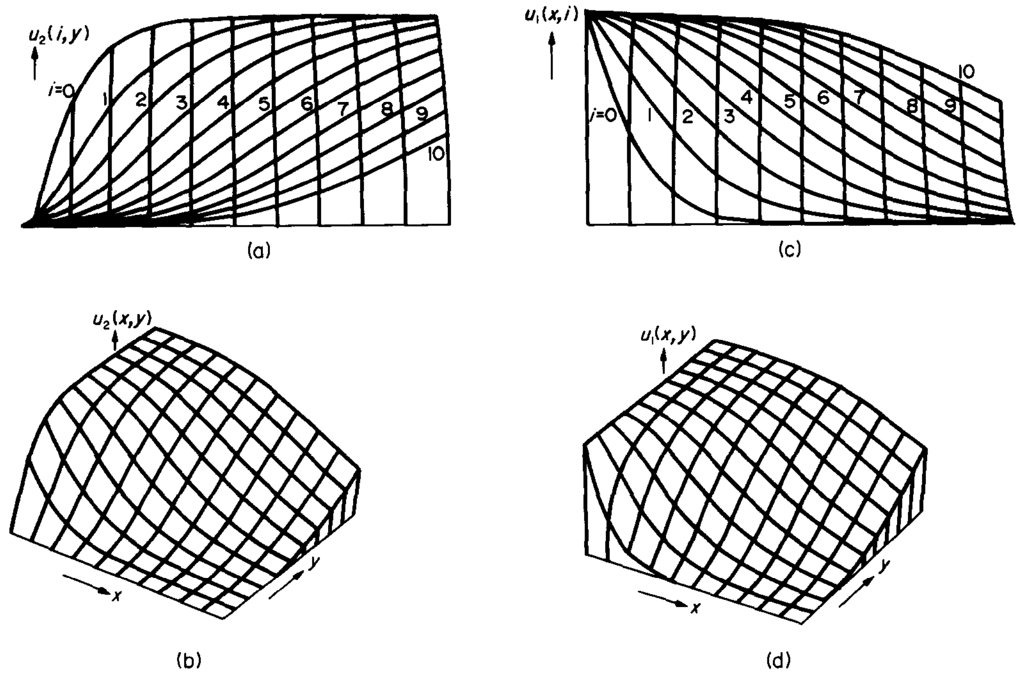


FIG. 2. Temperature distributions of the stronger (a, b) and weaker (c, d) fluids.

6. These figures illustrate the heat exchanger effectiveness for the steady periodic operation and the nonuniform temperature distribution for period 1 inlet fluid shown in Fig. 7. The numerical analysis was performed for $\beta_s = 0.25\beta_1$, $N_{tu1} = N_{tu2}$, $C_1 = C_2$, $\delta = 0.25$ (Fig. 5), $\delta = 0.5$ (Fig. 6) and the $50 \times 50$ subintervals for $x, t$ for any pair of $(C, N_{tu})$. The obtained results confirm the facts stated by Brandemuehl and Banks [7].

(a) For $C \geqslant 5$ the nonuniformities of the inlet temperatures have very little effect on effectiveness.

(b) The difference between effectiveness for nonuniform and uniform inlet temperatures increases as $\delta$ increases. The equilibrium theory ($N_{tu} \to \infty$) predicts effectiveness $\varepsilon_n \to 1$ for $C_n > 1$. This fact is confirmed in Table 1 of the paper by Brandemuehl and Banks [7] but their analytical solution shown in Fig. 5 [7] is not as accurate as the results obtained via model (30) which are shown in Figs. 5 and 6.

## 3. FURTHER APPLICATIONS OF THE 2-D DISCRETE MODELS

As stated in the Introduction the theory of 2-D systems furnishes other discrete state-space models which can also be used for the analysis of various

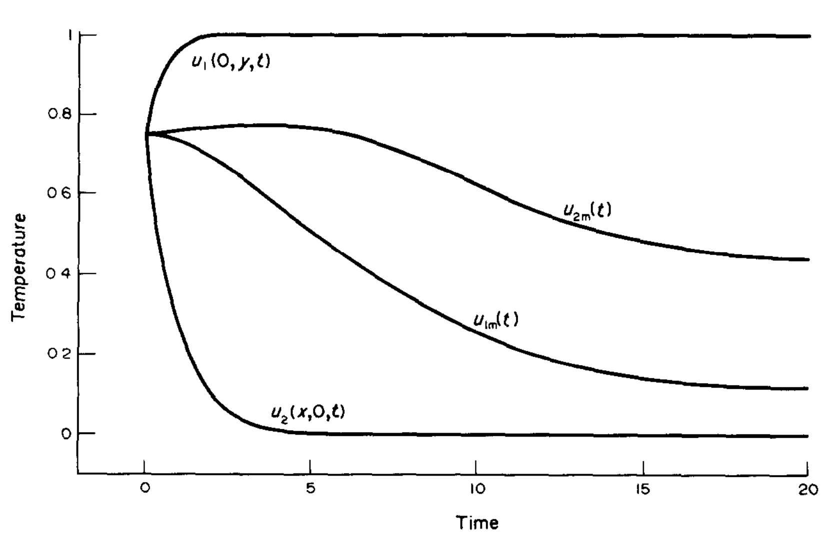


FIG. 3. Mean mixed outlet and inlet fluid temperatures of a cross-flow heat exchanger.

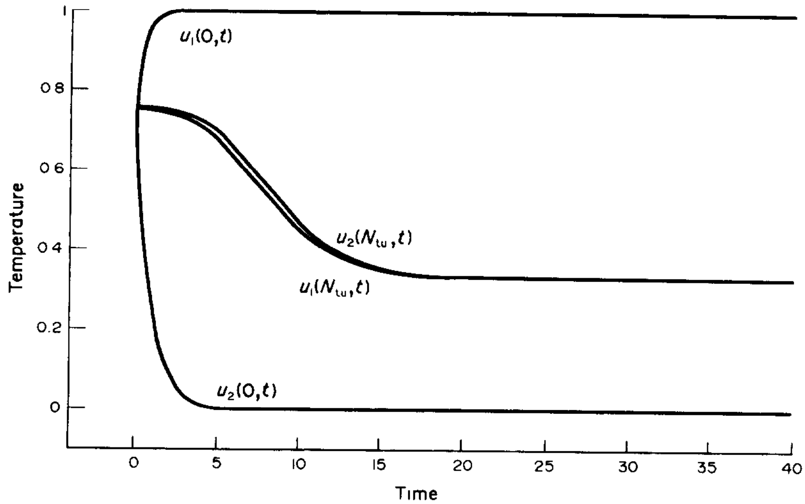


FIG. 4. Outlet temperatures for a parallel heat exchanger.

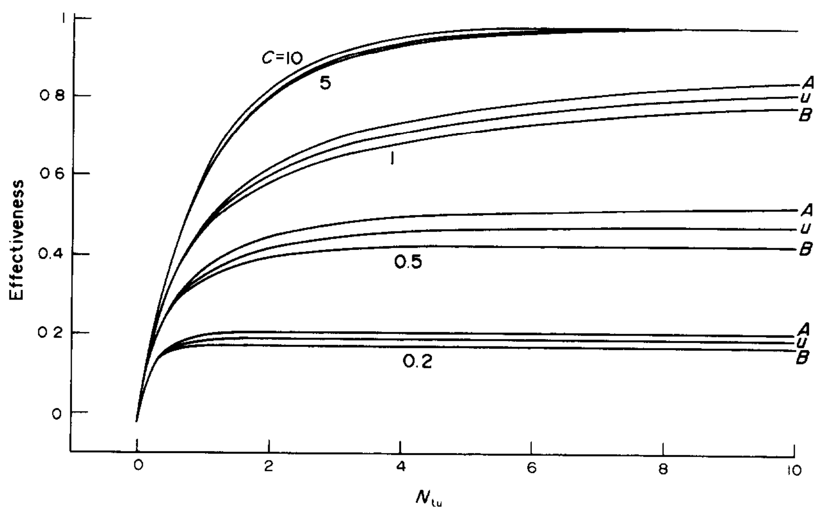


FIG. 5. Effectiveness of a rotary heat exchanger for $\delta = 0.25$.

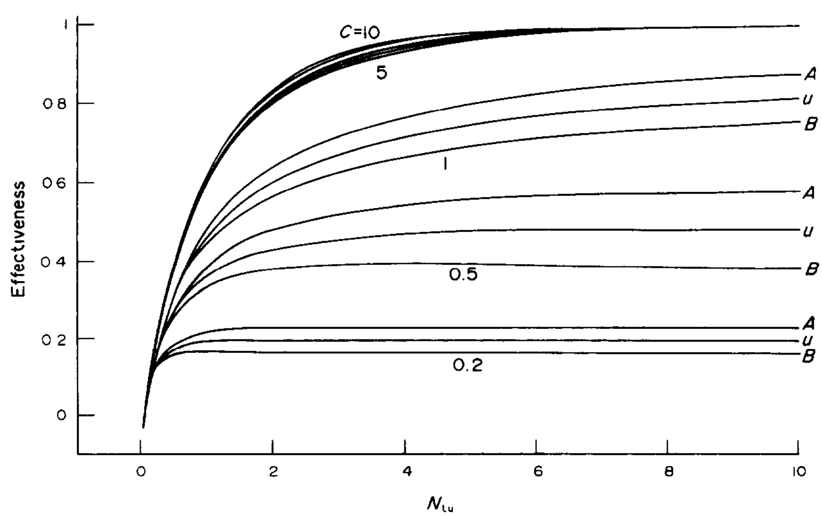


FIG. 6. Effectiveness of a rotary heat exchanger for $\delta = 0.5$.

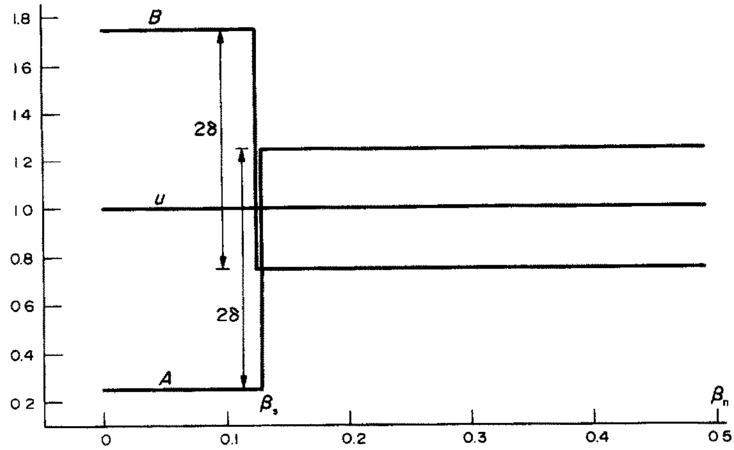


FIG. 7. Temperature distribution of period 1 inlet fluid.

partial differential equations. Among others the first and second models of Fornasini and Marchesini [15] are perhaps the best known.

If, for example we assume that there exist first derivatives of $f(x, y)$ in equation (1) with respect to $x$ and $y$, then simple manipulations yield from equation (1)

$$\frac{\partial^2 u_1}{\partial x \partial y} - a_{11}\frac{\partial u_1}{\partial y} - a_{22}\frac{\partial u_1}{\partial x} - (a_{21}a_{12} - a_{11}a_{22})u_1 = v \tag{31}$$

where

$$v = b_1 \frac{\partial f}{\partial y} - (a_{22}b_1 - a_{12}b_2) f$$

and

$$u_1(0, y) = \varphi_1(y)$$
$$u_1(x, 0) = \exp(a_{11}x)\varphi_1(0) + \int_0^x \exp[a_{11}(x-\tau)][a_{12}\varphi_2(\tau) + b_1 f(\tau, 0)]\,d\tau.$$

Replacing the mixed derivative in equation (31) with

$$\frac{\partial^2 u_1}{\partial x \partial y} \simeq \frac{u_1(i+1,j+1) - u_1(i+1,j) - u_1(i,j+1) + u_1(i,j)}{\Delta x\,\Delta y}$$

and the first-order derivatives with the forward difference quotients, we obtain

$$\begin{aligned} u_1(i+1,j+1) &= [1 + a_{11}\Delta x]u_1(i,j+1) \\ &\quad + [1 + a_{22}\Delta y]u_1(i+1,j) \\ &\quad - [1 + a_{11}\Delta x + a_{22}\Delta y \\ &\quad + (a_{21}a_{12} - a_{11}a_{22})\Delta x\,\Delta y]u_1(i,j) \\ &\quad + \Delta x\,\Delta y\,v(i,j) \\ &\equiv A_1 u_1(i,j+1) + A_2 u_1(i+1,j) \\ &\quad + A_0 u_1(i,j) + B f(i,j) \end{aligned} \tag{32a}$$

where

$$\begin{aligned} &A_1 = 1 + a_{11}\Delta x,\ A_2 = 1 + a_{22}\Delta y,\ B = \Delta x\,\Delta y \\ &A_0 = -1 - a_{11}\Delta x - a_{22}\Delta y - (a_{21}a_{12} \\ &\qquad - a_{11}a_{22})\Delta x\,\Delta y,\ f(i,j) = v(i,j). \end{aligned} \tag{32b}$$

Suppose that the parameters $a_{11}$, $a_{12}$, $a_{21}$ and $a_{22}$ depend on $x$ and $y$, or equivalently on $i$ and $j$, then from equation (31) the following shift-varying equation is obtained:

$$\begin{aligned} u_1(i+1,j+1) &= A_1(i,j+1)u_1(i,j+1) \\ &\quad + A_2(i+1,j)u_1(i+1,j) + A_0(i,j)u_1(i,j) + v(i,j) \end{aligned} \tag{33}$$

where

$$A_1(i,j+1) \equiv 1 + \Delta x\,a_{11}(i,j+1)$$

$$\begin{aligned} A_2(i+1,j) \equiv [&a_{12}(i,j+1) \\ &+ \Delta y\,a_{22}(i,j)a_{12}(i,j+1)]/a_{12}(i,j) \end{aligned}$$

$$\begin{aligned} A_0(i,j) \equiv [&-a_{12}(i,j+1) - a_{11}(i,j)a_{12}(i,j+1)\Delta x \\ &- a_{21}(i,j)a_{12}(i,j+1)a_{12}(i,j)\,\Delta x\,\Delta y \\ &- a_{22}(i,j)a_{12}(i,j+1)\Delta y \\ &+ a_{22}(i,j)a_{12}(i,j+1)a_{11}(i,j)\Delta x\,\Delta y]/a_{12}(i,j) \end{aligned}$$

$$\begin{aligned} v(i,j) \equiv\ &B_1\Delta x\,f(i,j+1) \\ &- B_1\Delta x\,a_{12}(i,j+1)f(i,j)/a_{12}(i,j) \\ &- a_{12}(i,j+1)a_{22}(i,j)\Delta x\,\Delta y\,B_1 f(i,j)/a_{12}(i,j) \\ &+ B_2\Delta x\,\Delta y\,a_{12}(i,j+1)f(i,j). \end{aligned} \tag{34}$$

The solution of equation (34) for given boundary conditions $u_1(i, 0)$ and $u_1(0, j)$ takes the form

$$u_1(i,j) = \sum_{k=1}^{i} \{T_{i,j}^{i-k,j-1} A_2(k,0) u_1(k,0) + T_{i,j}^{i-k-1,j-1}[A_0(k,0)u_1(k,0)+f(k,0)]\} + \sum_{l=1}^{j} \{T_{i,j}^{i-1,j-l} A_1(0,l) u_1(0,l) + T_{i,j}^{i-1,j-l-1}[A_0(0,l)u_1(0,l)+f(0,l)]\} + T_{i,j}^{i-1,j-1}[A_0(0,0)u_1(0,0)+f(0,0)] + \sum_{k=1}^{i}\sum_{l=1}^{j} T_{i,j}^{i-k-1,j-l-1} f(k,l) \quad (35)$$

where $T_{i,j}^{k,l}$ is defined recursively as follows:

$$T_{i,j}^{0,0} = I \quad \text{for} \quad i,j \geq 0$$

$$T_{i,j}^{k,l} = A_0(i-1,j-1)T_{i-1,j-1}^{k-1,l-1} + A_1(i-1,j)T_{i-1,j}^{k-1,l} + A_2(i,j-1)T_{i,j-1}^{k,l-1} \quad \text{for } i,j,k,l = 0,1,2,\ldots$$

$$T_{i,j}^{k,l} = 0 \text{ for } i<0 \text{ or } j<0 \text{ or } k<0 \text{ or } l<0. \quad (36)$$

Various dynamical properties of the above models have been studied [18, 35, 36, 38].

## 4. SINGULAR 2-D DISCRETE MODELS

Although the compact formulas (4) for the 2-D discrete systems show a similarity with 1-D discrete-time systems [40, 41] it is known that some of the distributed parameter systems have no natural notions of causality in one or more variables. Non-causal 2-D discrete systems can be realized by so-called singular or generalized 2-D discrete models [26, 33, 34, 39].

A natural extension of the 1-D singular discrete model is

$$\begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix} \begin{bmatrix} u_1(i+1,j) \\ u_2(i,j+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} u_1(i,j) \\ u_2(i,j) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} f(i,j) \quad (37)$$

or in a more compact form

$$Eu' = Au + Bf \quad (38)$$

where $E$ is a constant singular matrix ($\det E = 0$).

The following necessary and sufficient condition given by Lewis and Mertzios [28] should be satisfied for the unique solution of (38):

$$\operatorname{rank}(EZ-A, B) = \operatorname{rank}(EZ-A) \quad (39a)$$

where

$$Z = \begin{bmatrix} z_1 I_{n_1} & 0 \\ 0 & z_2 I_{n_2} \end{bmatrix}. \quad (39b)$$

On the other hand, if the solution exists, it is unique with respect to $u(z_1, z_2)$ if and only if

$$\operatorname{rank}(EZ-A) = n_1 + n_2. \quad (40)$$

If condition (40) is satisfied we call the matrix pencil $(EZ-A)$ or equivalently system (38), regular.

Due to singularity of matrix $E$, the transfer function of equation (37) together with the output equation

$$y(i,j) = Cu(i,j) + Df(i,j) \quad (41)$$

is not causal for one or both variables. This means that the solution $u(i,j)$ may depend on the boundary conditions $u_1(0,k)$, $u_2(l,0)$ and $f(k,l)$ for $k > i$ and/or $l > j$. The so-called index of the nilpotency is of important interest here [28]. It shows that some relations between the components of $u_1$ and $u_2$ in equation (37) are not dynamical but algebraic.

The first natural application of the 2-D singular systems is for the well-known heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + bf \quad (42)$$

with given boundary and initial conditions $u(0,t)$ and $u(x,0)$. A simple explicit approximation of equation (42) yields for $\Delta t/\Delta x \to 0$

$$u(i,j+1) = cu(i+1,j) + (1-2c)u(i,j) + cu(i-1,j) + bf(i,j). \quad (43)$$

It is easy to notice that equation (43) is causal in the $j$ (time) and noncausal in the $i$ (distance) directions. Defining new variables

$$z(i,j) = cu(i,j) + u(i-1,j)$$
$$v(i,j) = u(i-1,j) \quad (44)$$

we arrive at the following 2-D singular equation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v(i+1,j) \\ z(i+1,j) \\ u(i,j+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -c & c(2+c) \\ -1 & 1 & -c \end{bmatrix} \begin{bmatrix} v(i,j) \\ z(i,j) \\ u(i,j) \end{bmatrix} - \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} f(i,j). \quad (45)$$

Equation (45) and the output equation (41) for $C = [0 \ 0 \ 1]$, $D = 0$ give the following transfer function:

$$Y(z_1,z_2)/F(z_1,z_2) = C(EZ-A)^{-1}B = \frac{bz_1}{z_1^2 c - z_1 z_2 - z_1(1+2c) + c} \quad (46)$$

which is of course identical to the transfer function obtained directly from equation (43).

It is worth pointing out that an efficient recursive algorithm given by Lewis and Mertzios [28] can be used for computation of any 2-D transfer function and its 2-D fundamental matrix sequence. Having obtained the 2-D fundamental matrix sequence one is in position to write a general response formula for system (45) [33].

Also some additional problems such as identification, optimal control, inverse of 2-D singular systems, etc. can be solved. Further work in this direction is now in progress.

## 5. CONCLUSIONS

We have shown in this paper that the 2-D discrete state-space and singular equations can be used to solve some problems of the heat transfer analysis. Making use of the well-known finite difference approximations to the continuous models of the cross-flow, parallel-flow and rotary heat exchangers we were able to study their properties, transient responses and effectiveness via 2-D Roesser and Fornasini–Marchesini models.

The forward difference quotients used in our technique are not the only possible approximations. The well-known bilinear transformation for the 2-D systems can also be applied to yield another 2-D Roesser model. For example, it is a straightforward exercise to apply the results of Lodge and Fahmy [29] for another proof of equality (10).

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## ECHANGEURS DE CHALEUR ET THEORIE DU TRAITEMENT LINEAIRE D'IMAGE

**Résumé**—On montre que l'analyse transitoire de quelques échangeurs de chaleur peut être développée avec des équations linéaires de la théorie du traitement d'image. Les équations aux dérivées partielles des échangeurs à courants croisés, à courants parallèles et échangeurs rotatifs sont considérées avec les modèles discrets correspondants pour le traitement linéaire d'image. Quelques exemples numériques montrent que la nature des problèmes de transfert de chaleur ou de masse est semblable à celle du traitement d'image.

## WÄRMEAUSTAUSCHER UND DIE LINEARE BILDVERARBEITUNGSTHEORIE

**Zusammenfassung**—In dieser Arbeit wird gezeigt, daß die instationäre Analyse einiger Wärmeaustauschertypen auf einfache Art mit den linearen Gleichungen der Bildverarbeitungstheorie erfolgen kann. Partielle Differentialgleichungen für Kreuz- und Parallelstrom- sowie Rotationswärmeaustauscher werden zusammen mit den entsprechenden diskreten Modellen der linearen Bildverarbeitung behandelt. Einige Zahlenbeispiele zeigen, daß das Wesen von Wärme- und/oder Stoffübertragungsprozessen demjenigen der Bildverarbeitung ähnlich ist.

## ТЕПЛООБМЕННИКИ И ТЕОРИЯ ОБРАБОТКИ РЕЗУЛЬТАТОВ МЕТОДОМ ЛИНЕЙНЫХ ИЗОБРАЖЕНИЙ

**Аннотация**—Показано, что нестационарный анализ некоторых теплообменников может быть легко проведен на основе линейных уравнений теории изображений. Рассматриваются дифференциальные уравнения в частных производных для теплообменников с перекрестным, параллельным и вращающимся потоками, а также соответствующие дискретные модели для линейных изображений. Некоторые численные примеры показывают, что характер задач тепло- и/или массопереноса сходен с характером задач, решаемых методом изображений.